The Answer is $2^n \cdot n!$ What's the Question?

Gary Gordon

1. INTRODUCTION. Allison, Cory, Heidi, and Zach are relaxing in the math coffee room. Allison says, "Did you see Jeopardy last night? I'm always excited when they have *Mathematics* as a category, but their math questions are always so lame!" Cory and Zach agree with Allison. Heidi says "Why don't they ever have an answer like $2^n n!$? Then the contestants would have to find a problem with $2^n n!$ as the answer. That might not be so easy, even for Alex."

Cory says "I've got it: What do you get when you multiply 2^n and $n!$?"

Ignoring Cory's correct but uninteresting solution, Allison continues, "Since we're mathematicians, our goal should be to find really good questions, using all sorts of math. We should each find our own question, one that uses our own field. Remember, we're trying to find good questions having $2^n n!$ as the answer."

The rest of this paper is devoted to the approaches taken by these four (fictitious, but aptly named) mathematicians. These approaches are from three different fields: algebra, geometry and combinatorics. This allows us to see the same problem from four different viewpoints, with each successive explanation encompassing the previous ones.

The algebraic and geometric approaches given here are well known, though perhaps not as widely known as they should be. The combinatorial approach is recent, incorporating a graph theory algorithm to produce the desired connections among these viewpoints.

In the hope of making this note more readable, the proofs are frequently sketched broadly. We hope enough details remain for the interested reader to fill in any missing steps (or look them up).

2. ALLISON'S TURN. Allison's specialty is algebra; she decides to find a group with $2^n n!$ elements. She needn't look far; the full symmetry group of an n-dimensional hypercube will do. There are n^2 mirror symmetry hyperplanes for an n-dimensional hypercube. One way to see this is to place the 2^n vertices of the hypercube at the points $(\pm 1, \dots, \pm 1)$. Then the reflections of the hypercube occur in the hyperplanes whose equations are:

- $x_i = 0$ (the n coordinate hyperplanes),
- $x_i = x_j$ (there are $\binom{n}{2}$ of these), and
- $x_i = -x_j$ (there are $\binom{n}{2}$ of these, too),

where $1 \le i < j \le n$. See Figure 1 for a picture of the cube and its nine planes of reflection. In this figure, $x_i = 0$ (for $1 \le i \le 3$) corresponds to a plane that slices through the middle of four faces of the cube, while $x_i = x_j$ and $x_i = -x_j$ $(i \ne j)$ correspond to planes that contain the diagonals of two opposite faces and two edges of the cube.

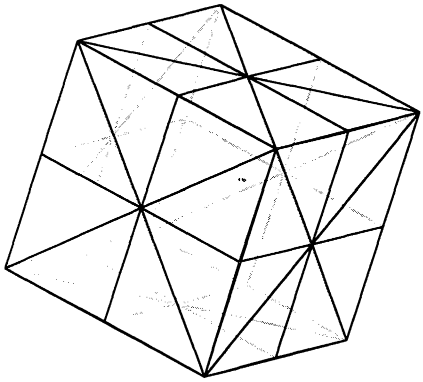

Figure 1. A cube and its planes of symmetry.

Let H_n denote the n-dimensional hypercube and let G_n be its symmetry group. Allison needs to show there are precisely $2^n n!$ elements in G_n. She decides to give an inductive argument. When $n = 1$, the hypercube is just a line segment, which has symmetry group $\mathbf{Z}_2$. In general, H_n has $2n$ *facets* (faces of dimension $n - 1$, each of which is the hypercube H_{n-1}). Allison constructs an arbitrary symmetry of H_n as follows:

1. Choose one of the $2n$ facets of H_n and move the hypercube so that this facet is on the bottom; the bottom of the hypercube is the facet that meets the x_n-axis at the point $(0, \ldots, 0, -1)$.
2. Use an element of G_{n-1} as a symmetry of this bottom face; in general, this symmetry permutes the other facets of H_n.

In 3 dimensions, this corresponds to picking one of the 6 faces of the cube for the bottom, then using one of the 8 symmetries of the square on this bottom face. This gives 48 symmetries, only 24 of which can be realized by rigid motions. The remaining symmetries are either reflections (there are 9, as we have already seen) or *rotary reflections*, i.e., reflections followed by rotations (there are evidently 15 of these). It is an interesting exercise to find explicit descriptions of these 15 rotary reflections for the cube.

Every symmetry of H_n can be obtained in this manner, since any symmetry must carry facets to facets. Then $|G_n| = 2n|G_{n-1}|$, which, together with the initial condition $|G_1| = 2$, gives $|G_n| = 2^n n!$

Allison's algebraic solution. Let G_n be the symmetry group of an n-dimensional hypercube. Then the order of G_n equals $2^n n!$

This approach gives us our answer in one way, but gives us very little information about the structure of the group G_n. We investigate the group in a bit more detail in the concluding section.

3. HEIDI'S TURN. Hyperplane arrangements form the background for the following classic problem, a favorite in mathematics contests:

- What is the largest number of regions produced when n lines are drawn in the plane?

This problem is also useful when introducing mathematical induction. The 3-dimensional version appeared as MONTHLY Problem E554 [**8**], where J. L. Woodbridge of Philadelphia asked:

- Show that n cuts can divide a cheese into as many as $(n+1)(n^2-n+6)/6$ pieces.

Both of these questions are answered by a general formula discovered by L. Schläfli (published posthumously in 1901):

- The largest number of regions produced when n hyperplanes are drawn in d-dimensional space equals $\sum_{k=0}^{d}\binom{n}{k}$.

Counting the regions of various hyperplane arrangements is the beginning of a beautiful subject, with deep ties to algebra, topology, and combinatorics. A classic reference is [**7**]. Heidi has studied hyperplane arrangements and thinks she can use them to construct a good question. Since the algebraic approach has a strong geometric flavor, her first solution is to copy Allison's solution, without using groups. Recall that Allison used hyperplanes to understand the group G_n. If Heidi simply uses the same collection of hyperplanes Allison used (without even mentioning the hypercube), she will get a dissection of space into open n-dimensional regions. How many regions are produced by this hyperplane arrangement?

Thus, Heidi is concerned with counting the regions of the hyperplane arrangement whose equations are $x_i = 0$, $x_i = x_j$, and $x_i = -x_j$. Heidi needs to show that the number of regions is $2^n n!$.

What do these regions look like? Heidi finds it is easier to imagine the regions by intersecting them with a cube, as in Figure 1. Then a typical region is formed as follows: Let $O = (0,0,0)$ be the center of the cube, let $P_1 = (1,0,0)$ be the center of a face, let $P_2 = (1,1,0)$ be the center of an edge adjacent to this face, and finally let $P_3 = (1,1,1)$ be a vertex adjacent to this edge. Then the region is the tetrahedron whose vertices are the four points O, P_1, P_2, and P_3. There 6 choices for P_1, 4 choices for P_2, and 2 choices for P_3, giving us 48 regions.

In general, Heidi produces a region by picking the center of the hypercube, then picking the center of one of its $2n$ facets, then picking one of the $2(n-1)$ faces (of dimension $n-2$) of the chosen facet, and so on. At stage k, she chooses the center of one of the $2(n-k)$ facets surrounding an $n-k$–dimensional face of the hyperplane. The $n+1$ points produced are the vertices of a simplex; this simplex is the intersection of a region of the hyperplane arrangement with the hypercube. See [2, §7.6] for more on this approach.

The number of regions of the arrangement is just $2n \cdot 2(n-1) \cdots 4 \cdot 2 = 2^n n!$, so Heidi proudly announces

Heidi's hyperplane solution. Let A_n be the hyperplane arrangement given above. Then A_n decomposes space into $2^n n!$ regions of dimension n.

Heidi also wants to relate her approach to Allison's approach. She does so by exhibiting a one-to-one correspondence between the regions of the arrangement and the elements of the symmetry group G_n. To see this correspondence, first note that the symmetry group G_n acts on the regions of the arrangement. Now given any pair of regions A and B in her hyperplane arrangement, there is a unique element

$\sigma_{A,B} \in G_n$ with $\sigma_{A,B}(A) = B$ (G_n acts transitively on the arrangement). The existence of $\sigma_{A,B}$ follows because the regions themselves have no non-trivial symmetries–they are completely asymmetric. Then labeling an arbitrary region of the arrangement by the identity I of the group, Heidi gets her bijective correspondence between the regions $A_1, A_2, \ldots$ and the elements $\sigma_{I,A_1}, \sigma_{I,A_2}, \ldots$ Heidi has constructed a portion of the Cayley graph of G_n—the arrangement itself is equivalent to the geometric dual of the Cayley graph. See **[2]** for more on Cayley graphs.

4. ZACH'S TURN. If M is any $r \times k$ matrix, then the zonotope $Z(M)$ is a polytope in $\mathscr{R}^r$ given by

$$Z(M) = \left\{ \sum_{i=1}^{k} \lambda_i \mathbf{c}_i : -1 \le \lambda_i \le 1 \right\},$$

where $\mathbf{c}_i$ is the i^{th} column of M. Since the $\mathbf{c}_i$ are just an arbitrary collection of vectors in $\mathscr{R}^r$, the study of zonotopes is very natural in geometry or linear algebra. It is also fun to build 3-dimensional models of them using a good kit; Polydrons and Zometools are both good choices.

The hypercube H_n is a product of n intervals: $H_n = [-1,1] \times \cdots \times [-1,1]$. Zonotopes are projections of hypercubes; the definition shows how each interval $[-1,1]$ appears. The name *zonotope* arises from the fact that the facets determine 'zones' in space. Zonotopes are an important and well-studied class of polytopes with applications to oriented matroids, tiling problems and more. See **[1]**, **[2]**, and **[14]** for interesting connections among zonotopes, groups, geometry and oriented matroids.

Zach knows zonotopes; his idea is to find a class of zonotopes in which $2^n n!$ counts something. An obvious choice for a matrix whose columns generate the zonotope (given the first two solutions) is the matrix of normal vectors of the hyperplanes considered previously; $x_i = 0$, $x_i = x_j$, $x_i = -x_j$. For $n = 3$, a picture of the zonotope appears at the bottom of Figure 2. This polytope is a *rhombitruncated cuboctahedron*, one of the Archimedean solids. This zonotope arises as a truncation of a cube; the vertices of the cube correspond to the hexagons of the zonotope, the edges of the cube correspond to the squares and the faces (squares) of the cube correspond to the octagons. See Figure 2 for an illustration of how the truncation process produces the zonotope.

Zach constructs the $n \times n^2$ matrix M_n whose column vectors are normal to the hyperplanes (yes, the same hyperplanes Heidi used), and he calls the associated zonotope $Z(M_n)$. In general, order the columns of M_n as in the following example:

$$M_3 = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & 1 & -1 \end{bmatrix}$$

Zach now just counts the number of vertices of the zonotope $Z(M_n)$. Superimposing a picture of $Z(M_3)$ and the hyperplane arrangement A_3, Zach realizes that there is a one-to-one correspondence between the vertices of $Z(M_3)$ and the regions of the arrangement (see Figure 3). Note that each vertex of $Z(M_3)$ appears in the center of a region of the arrangement. Generalizing from 3-dimensions to n-dimensions (frequently dangerous but also frequently productive), he guesses

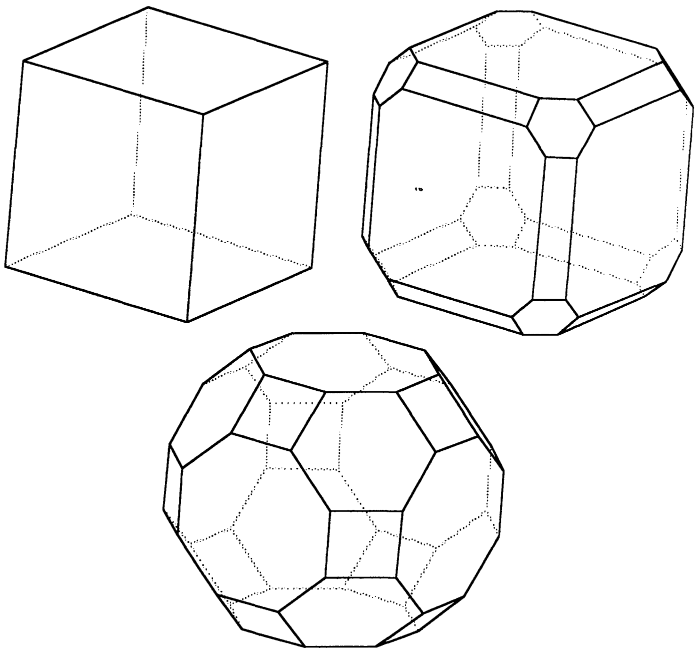

Figure 2. Truncating a cube to produce a zonotope.

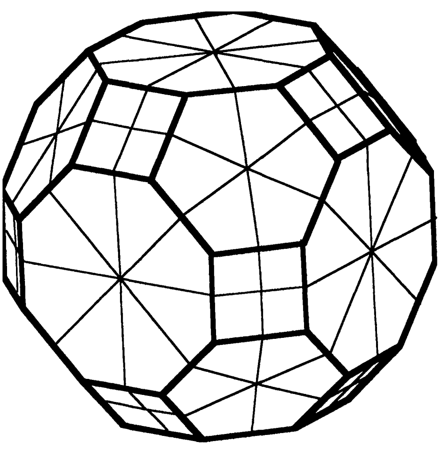

Figure 3. A zonotope and its planes of symmetry.

Zach's zonotope solution. The zonotope $Z(M_n)$ has $2^n n!$ vertices.

To see why this is true in general, Zach uses a little linear algebra. Here is a sketch of his ideas: First, he realizes that if the vector $\mathbf{v}$ in $\mathscr{R}^n$ is a vertex of the zonotope, then $\mathbf{v} = M_n\mathbf{p}$, where $\mathbf{p}$ is a column vector, each of whose n^2 entries equals 1 or -1. This follows from remembering that $Z(M_n) = \Sigma_{i=1}^{n^2} \lambda_i \mathbf{c}_i$, where $-1 \leq \lambda_i \leq 1$; a vertex of $Z(M_n)$ can ensue only when $\lambda_i = 1$ or -1. There are 2^{n^2} such ± 1 vectors $\mathbf{p}$, representing the vertices of an n^2-dimensional hypercube, the hypercube that is projected to $Z(M_n)$.

Each of these 2^{n^2} vertices of the hypercube produces a vector $\mathbf{v}$ that is now a possible vertex of $Z(M_n)$. Which $\mathbf{v}$ are the zonotope vertices? Zach figures out that any vector whose entries are some permutation of $\{1a_1, 3a_2, 5a_3, \ldots, (2n-1)a_n\}$, where each a_i equals 1 or -1, arises as $M_n\mathbf{p}$ for some vector of ± 1's.

Why do the vertices look like this? Every row of the matrix M_n has exactly $2n-1$ non-zero entries. Thus any coordinate of $M_n\mathbf{p}$ is bounded above by $2n-1$ and below by $-(2n-1)$. Once $\mathbf{p}$ has been chosen to make a certain coordinate of $M_n\mathbf{p}$ equal to $\pm(2n-1)$, the next largest (or smallest) coordinate value possible is $\pm(2n-3)$. The argument proceeds inductively; see **[3]** for a complete proof.

Conversely, each such $\mathbf{v}$ must be a vertex of the zonotope, since no such vector is a convex combination of other points in $Z(M_n)$. In particular, the matrix product $M_n\mathbf{p}$ can never produce any entry larger than $2n-1$ in absolute value. Thus, if $\pm(2n-1)$ appears as an entry in $\mathbf{v}$, then $\mathbf{v}$ could not be a non-trivial convex combination of other vertices of the zonotope. Finally, no other $\mathbf{v}$ is a vertex of $Z(M_n)$, since no other $\mathbf{v}$ contains $\pm(2n-1)$ as an entry.

The key for us is simply the following: since there are $n!$ permutations of $[1, 3, \ldots, (2n-1)]$ and 2^n possible ways to assign ± 1 to each entry, Zach has produced $2^n n!$, as required.

5. CORY'S TURN. Zach did his job, but he ducked an important question in considering the matrix products $M_n\mathbf{p}$:

Question 1. Of the 2^{n^2} possible sign vectors $\mathbf{p}$, which ones produce the $2^n n!$ vertices of the zonotope $Z(M_n)$?

The job of tying all the pieces together falls to Cory, our combinatorialist. Cory realizes that finding a question whose answer is $2^n n!$ is not much of a challenge; he could just find all permutations of $[\pm 1, \pm 3, \ldots, \pm (2n-1)]$ (as in Section 4) and be done. This involves combinatorics only superficially, however; he seeks a deeper connection, as do we.

The matrix M_n that Zach used reminds Cory of incidence matrices. Remember that every column of M_n has either exactly one non-zero entry (which equals 1) or exactly two non-zero entries (which either are both equal to 1 or have one entry equal to 1 and the other equal to -1). Cory decides to create a graph that has this matrix as its vertex-edge incidence matrix. Here's how the graph is defined: First, he labels n vertices of the graph he (modestly) calls C_n with the numbers $1, \ldots, n$ (corresponding to the rows of the matrix). Then the edges of the graph are filled in as follows:

1. Put a loop at every vertex (corresponding to the columns having one 1 and $n-1$ 0's),
2. Put an edge between every pair of vertices (corresponding to the columns having two 1's and $n-2$ 0's),
3. For each i and j such that $1 \le i < j \le n$, put a directed edge pointing from vertex i to vertex j (corresponding to the column having a 1 in position i, a -1 in position j and $n-2$ 0's).

Technically, C_n is a *mixed graph* because it mixes directed and undirected edges.

Cory has talked to Zach and knows about the connection between the matrix M_n and the zonotope $Z(M_n)$. In order to connect the graph C_n with the zonotope, he needs to interpret $M_n\mathbf{p}$, where $\mathbf{p} = [e_1, \ldots, e_{n^2}]^T$ and each $e_i = \pm 1$. He decides to use the entries of $\mathbf{p}$ to label some of the edges of C_n with ± 1. If column k of

M_n corresponds to an undirected edge in C_n, Cory labels this edge with e_k, which equals either 1 or -1. Similarly, if column k corresponds to a loop in C_n, he labels the loop with e_k. Finally, if column k of M_n corresponds to a directed edge from vertex i to vertex j, he leaves this directed edge unchanged if $e_k = 1$ and he reverses the direction if $e_k = -1$.

This process produces a *signed, oriented mixed graph* $C_n(\mathbf{p})$. See Figure 4 for an example with M_3 and the vector $\mathbf{p} = [1, -1, 1, -1, 1, 1, -1, 1, -1]^T$. To understand how this $\mathbf{p}$ produces the signs and arrows shown, recall the order of the

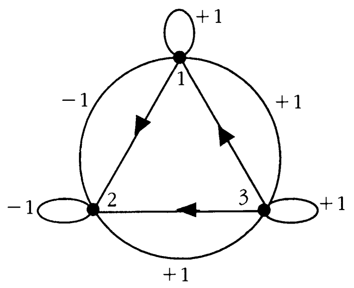

Figure 4. Signed mixed graph $G_3(\mathbf{p})$.

normal vectors appearing as the columns of M_3: The first three columns correspond to the loops at vertices 1, 2, and 3. Columns 4, 6, and 8 correspond to the (undirected) edges joining vertices 1 and 2, 1 and 3, and 2 and 3, respectively. Thus, for example, since $e_4 = -1$, the edge joining vertices 1 and 2 is signed with -1. Columns 5, 7, and 9 correspond to the directed edges joining vertices 1 and 2, 1 and 3, and 2 and 3, respectively. Thus, for example, since $e_7 = -1$ in the example, the edge directed from vertex 1 to vertex 3 is reversed.

There is a one-to-one correspondence between signed, oriented mixed graphs and the sign vectors $\mathbf{p}$. It is interesting to note that these sign vectors also arise naturally in the context of the hyperplane arrangements—Heidi's hyperplane approach to the problem. For any region R in the hyperplane arrangement, let v be a point in R. Thinking of v as a vector, compute the usual inner product of v with each of the n^2 normal vectors. Recording only whether these inner products are positive or negative produces a sign vector of length n^2; this simply records which side of each hyperplane the point v lies. Among all possible 2^{n^2} potential sign vectors that could arise in this way, it turns out that only $2^n n!$ do arise. These are precisely the sign vectors that produce vertices of the associated zonotope.

As anyone who has ever put up wallpaper knows, it is easy to move a bubble beneath the paper from place to place, but it's hard to get rid of the bubble. Cory has now shifted the problem of determining the $2^n n!$ vertices of the zonotope (or the regions of the hyperplane arrangement) to a graph theory problem.

Question 2. Of the 2^{n^2} possible oriented, signed mixed graphs $C_n(\mathbf{p})$, which ones produce the $2^n n!$ vertices of the zonotope $Z(M_n)$?

Cory's goal is to separate the two factors 2^n and $n!$ by having each one count a particular action. Now there are two aspects to $C_n(\mathbf{p})$: the orientation of the directed edges and the signing of the undirected edges. Cory decides to call a sign vector $\mathbf{p}$ *good* if it corresponds to a vertex of the zonotope. Cory notices a few things about these good $\mathbf{p}$: First, the orientation of $C_n(\mathbf{p})$ is *acyclic* when $\mathbf{p}$ is good.

This means that the vertices can be linearly ordered $v_1, v_2, \ldots, v_n$ so that every directed edge adjacent to v_1 points away from v_1, every directed edge adjacent to v_2 (except for the edge from v_1 to v_2) points away from v_2, and so on. In the language of tournaments, the directed edges of $C_n(\mathbf{p})$ form a *transitive* tournament —if a beats b and b beats c, then a beats c. There are $n!$ acyclic orientations—Cory is half-way home.

The second important point Cory realizes has to do with the signs on the undirected edges. When $\mathbf{p}$ is good, these signs can be obtained 'vertex by vertex' as follows: Assume $v_1, v_2, \ldots, v_n$ is the linear order and let $\mathbf{b} = [b_1, \ldots, b_n]^T$ be one of the 2^n vectors of 1's and -1's. If $b_1 = 1$, then he signs with a 1 all undirected edges incident to v_1 (including the loop); if $b_1 = -1$, then all undirected edges (including the loop) incident to v_n are signed with a -1. Now Cory removes v_1 or v_n from the list (depending on whether $b_1 = 1$ or -1), giving a new list of $n - 1$ vertices. He then repeats the process for b_2: if $b_2 = 1$, then he signs with a 1 all undirected edges incident to the first vertex on the new list (v_1 when $b_1 = -1$ and v_2 when $b_1 = 1$) (including the loop) *that have not previously been signed*; if $b_1 = -1$, then he signs with a -1 all undirected edges (including the loop) incident to the last vertex on the list (v_n when $b_1 = 1$ and v_{n-1} when $b_1 = -1$) *that have not previously been signed*. The process continues until all of the signs $b_1 - b_n$ have been processed. Since there are 2^n sign vectors $\mathbf{b}$ and each one gives a unique signing, Cory has the 2^n factor directly visible, too.

For example, if $\mathbf{b} = [1, -1, 1]^T$ and $v_1 = 3, v_2 = 1$, and $v_3 = 2$, he first paints the edges adjacent to vertex 3 with 1's, then paints the previously unsigned edges adjacent to 2 with -1's and finally paints the remaining unsigned edges adjacent to 1 with a 1; the only edge painted in this last step is the loop at vertex 1. See Figure 4.

Cory gives an inductive argument to show why this works; you can see it in [**3**]. The key for us is that this procedure for producing an oriented signed graph gives a direct link to our $2^n n!$ problem family: Since there are $n!$ orderings of the n vertices (to produce an acyclic orientation) and there are 2^n sign vectors $\mathbf{b}$ of length n, Cory immediately gets the answer of $2^n n!$.

Cory's combinatorial solution. There are precisely $2^n n!$ oriented, signed mixed graphs $C_n(\mathbf{p})$ corresponding to good sign vectors $\mathbf{p}$.

6. REBUTTAL.

Allison, Heidi, Zach, and Cory meet in the coffee room to share their work. Allison sees a connection between the structure of the group G_n and the ubiquitous $2^n n!$

Allison's turn. Allison knows that the symmetry group G_n of a hypercube decomposes as a semi-direct product:

$$G_n = \mathbb{Z}_2^n \rtimes S_n.$$

She views the symmetry group of a hypercube as follows:

- First label the $2n$ facets of the hypercube by the symbols $1, 1^*, 2, 2^*, \ldots, n, n^*$, where the symbol i represents the facet contained in the hyperplane $x_i = 1$ and i^* represents the facet contained in the hyperplane $x_i = -1$.
- Choose a vertex v of the hypercube and record the ordered list of n facets incident to v. The starting point for the list can be determined uniquely by choosing the top or bottom facet first (whichever is incident to v) and fixing an orientation in space.

- Then an arbitrary symmetry of the hypercube can be broken down into two steps: First choose a permutation of the labels around the vertex v, then map v to some other vertex v'.

Allison explains to her colleagues: "The first step in this procedure can be accomplished by composing certain reflections through v (the collection of all reflections through v generates the symmetric group S_n), while the second step amounts to choosing an element from the normal subgroup $\mathbb{Z}_2^n$ to move from v to v' (accomplished by conjugating the permutation by the appropriate element of $\mathbb{Z}_2^n$, which is generated by the reflections perpendicular to the coordinate axes). Hey! Is anyone still awake?"

Cory's turn. Cory sees that the graph algorithm that produces the good sign vector **p** does (to a graph) more or less the same thing Allison just did (to the group G_n). To understand how the correspondence works, Cory reminds his colleagues that he first chooses an acyclic orientation of C_n, resulting in a permutation of the n vertices (which corresponds to the permutation of the n facets around the vertex v), then he picks a sign vector **b** of length n (which corresponds to mapping the vertex v in the hypercube to some other vertex). This gives a map of the $2^n n!$ elements of G_n to the collection of all $C_n(\mathbf{p})$ that are formed from good **p**'s.

Heidi adds, "It's really interesting how the four approaches used different areas of mathematics, but are so closely related. Let's send our solutions to Jeopardy!"

7. CONCLUDING REMARKS. The connections between hyperplane arrangements, zonotopes, and symmetry groups is explored in [**2**]. Further interpretations are examined in [**5**]. The sign vectors **p** considered here are maximal covectors in an associated oriented matroid [**1**]. The further connection with acyclic orientations of ordinary graphs is due to Curtis Greene [**4**], while the extension to mixed graphs appears in [**3**].

Signed graphs also provide a good way to understand the combinatorics of hyperplane arrangements. Tom Zaslavsky has developed a substantial theory for these combinatorial objects. See [**11**], [**12**], or [**13**] for a sample of this work. Many of his results generalize to other hyperplane arrangements. Zaslavsky has also considered hyperplane arrangements from a matroidal viewpoint. A very readable account appears in [**10**].

Many of the arguments given here can be rephrased using matrix groups; the reflections in the group G_n are easily represented by matrices. See [**2**] for more details on using linear algebra in this way.

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